JOURNAL OF NATURAL GEOMETRY, 2 (1992), No.2, 137-150. 0963-2654-92-2-7-137-150. US$14.00. UK£7.00

Asymptotology I: Problems, Ideas and Results

I. V. Andrianov and L. I. Manevich

Abstract. A heuristic description of the development of **asymptotology**, which is an old and useful branch of mathematical physics developing in a new way, is presented. The central idea is to simplify the model for a physical system by first studying its asymptotic behaviour and then modelling the actual behaviour by perturbing the asymptotic solution.

1. Introduction

Almost any physical theory formulated in mathematical terms in a general way is extremely complicated. Therefore, both in creating a theory and in its further development, the simplest limiting cases that admit analytical solutions are of paramount importance. It is quite common that in the limiting case there are fewer equations or the (differential) equation has a lower order or nonlinear equations are replaced by linear ones or the original system is subjected to a kind of averaging and so on and so forth.

Behind the above-mentioned idealizations, however diverse they may seem, lies a high degree of symmetry inherent in a mathematical model of the phenomenon at issue in its limiting situation. An asymptotic approach to a complex and perhaps "insoluble" problem consists basically of treating an original – insufficiently symmetric – system as approximating to a given symmetric one. It is basically important that the determination of corrections allows one to study deviations from the limiting case in a way which is much simpler than a direct study of the original system.

At first sight, the potentialities of such an approach are limited by a narrow range of variations in the parameters of the system.

1991 Mathematics Subject Classification: Primary: 03C99; 20F10
Secondary: 81V99

Experience gained in the study of various physical problems has shown, however, that in the case of system parameters varying considerably and the system itself departing from one limiting symmetric pattern, in general another limiting system, often with a less pronounced symmetry, exists and a perturbed solution can now be formed for the latter one.

This enables the system's behaviour to be defined over the entire range of the variations of the parameters using a finite number of limiting cases. Such an approach makes the most of one's physical intuition and contributes to its further enrichment and also leads to the formation of new physical concepts. Thus the boundary layer – an important concept in fluid mechanics – is of pronounced asymptotic nature and is related to the localization at the boundaries of a streamlined body in the zone where the viscosity of the fluid cannot be neglected. In the mechanics of a deformable rigid body and in the theory of electricity, similar phenomena are known as the *edge effect* and the *skin effect* respectively.

That the asymptotic method assists in relating different physical theories with one another is of little consequence. Albert Einstein would point out that "the happiest lot of a physical theory is to serve as a basis for a more general theory while remaining a limiting case thereof".

The above-mentioned problems will be clarified in this paper.

2. An illustrative example

As an illustration of the technical aspect of the method, consider a simple algebraic example. A biquadratic equation

$$x^4 - 2x^2 - 8 = 0 \tag{2.1}$$

is reduced to a quadratic one and readily solved by setting $z = x^2$. Then we have

$$x = \pm 2; \quad x = \pm\sqrt{2}\, j; \quad j = \sqrt{-1}.$$

Such a simplification is due to the symmetry of the equation: substituting $(-x)$ for x

does not change it. Let us assume that an original equation describes some given physical system with its parameters undergoing small changes and as a consequence the equation takes the form

$$y^4 - \epsilon y^3 - 2y^2 - 8 = 0 \tag{2.2}$$

In this case the system is said to have received a small perturbation, the expression $(-\epsilon x^3)$ is referred to as the "perturbation term" and ϵ as the "small parameter". The system becomes asymmetric, and the solution of the new equation can no longer be written in simple form. The new equations's roots y_i $(i = 1, \dots, 4)$, however, should not differ significantly from x_i, hence set $y_i = x_i$. The error of such a substitution is determined by the value of the discarded term $(-\epsilon x^3)$. To make the solution more accurate, let us represent it as a series

$$y_i = x_i + \epsilon y_i^{(1)} + \epsilon^2 y_i^{(2)} + \dots ; \qquad i = 1, \dots\dots, 4 \tag{2.3}$$

Substituting this expression into the perturbation equation and equating the coefficients of the same powers of ϵ we find

$$y_i = 0.25x_i^2 / (x_i^2 - 1); \quad i = 1, \dots\dots, 4. \tag{2.4}$$

Evaluation of corrections could be continued without any difficulty, but the deviation from the exact solution will inevitably increase with the increase in the value of ϵ.

Consider now the opposite case of big perturbations. Then the reciprocal ϵ^{-1} will be small. Then, the roots of equation (2.2) can be divided into two groups. As ϵ^{-1} tends to zero, three roots tend to zero and the fourth one increases indefinitely, the two groups lending themselves, as mentioned above, to expansions in the small parameter ϵ^{-1}.

$$y_1 = \epsilon + \dots\dots ; \tag{2.5}$$

$$y_2 = -2\epsilon^{-1/3} + \dots\dots ; \; y_{3,4} = \epsilon^{-1/3}(1 \pm \sqrt{3}\, j) \tag{2.6}$$

There exists, however, a region where the asymptotic approximation produces unsatisfactory results. This is the region where "small" ϵ's are already large and "large" ϵ's are still small. The problem of forming a solution within such a region on the basis of available limiting values is one of the most difficult ones when employing asymptotic methods as is the problem of deciding as to what is to be considered "small" or "large". This will be considered later.

Besides, it should be noted that perturbation solutions of a problem represented as expansions in series in the powers of a small parameter of type (2.3) do not necessarily converge to the solution which is being sought. The expansions are often asymptotic. A ratio of each term of the series to the preceding one tends to zero when the expansion parameter approaches its limiting value, say, zero; and the deviation of the sum of the first N terms of such a series from the function represented by the complete series is of the $(N + 1)$th order. (In examining a series for convergence, the parameter is regarded as fixed and the limit of the sum of N terms of the series is takes as N tends to infinity). In particular cases a divergent asymptotic series (with infinite limit) is sometimes more useful than a convergent one as only a few of the initial terms gives a fair approximation.

Let us consider some typical situations where the asymptotic approach is effective.

3. Reducing the dimensionality of a system

A high degree of an algebraic or a differential equation or a large number of such equations are all manifestations of one of the principal difficulties that arise in solving physical problems. This difficulty is sometimes called "the imprecation of dimensionality". In order to get over it, two antithetical approaches have been developed. The first one proves to be effective if individual elements of a system under consideration differ markedly from each other in one or another characteristic. Then by introducing characteristics of different elements – one is able to carry out an asymptotic reduction of dimensionality, or in other words, a reduction in the degree of freedom and then one can try to improve the solution obtained by using the asymptotic approximation. A typical example of such a situation is a three-body problem in celestial mechanics. The masses of celestial bodies (say, those of the Sun, the planet Jupiter and the Earth), as a rule, differ markedly, and a small parameter – mass ratio – enables as asymptotic reduction of the dimensionality. Based on this are the classical methods of celestial mechanics, a limiting (high symmetry) case being the exactly solvable two-body problem. Celestial mechanics is the first branch of science where the asymptotic method

(the perturbation theory) has played a dominant role, and moreover, this method was originally developed in response to the pressing necessity of solving the problem in celestial mechanics.

It should be noted that asymptotic methods are often used without being specifically regarded as such and even without being fully understood. Thus, one degree of freedom models are employed extensively in engineering. Clearly, employing such models always involves an asymptotic reduction in the dimensionality and the possibility, at any rate in principle, of finding the corresponding corrections, but a clear indication that this is the case is rare.

Let us now consider a second way of getting out of the difficulty.

4. Continualization

If a system under consideration consists of a set of homogeneous elements, then the asymptotic approach can be used not only for the reduction of dimensionality but also for increasing the dimensionality. Thus, we approach a highly important class of physical models where discrete systems are replaced by continuous ones.

As an example let us consider the longitudinal oscillations of an endless chain of the same masses connected by springs of equal length L and rate b. With the smooth oscillation form characterized by the displacement u_k at each point kL $(k = 0, \pm 1, \pm 2, \ldots)$, the chain can be replaced by a continuous rod, thus enabling us to change from the infinite system of ordinary differential equations

$$mu_{ktt} = c(u_{k+1} - 2u_k + u_{k-1})$$

to the single partial differential equation

$$mu_{tt} = cL^2 u_{xx}.$$

Degrees of freedom have now grown in number (the continuum replacing the countable set), and a relative simplicity of this limiting case of long-wave oscillations is due to the symmetry of the partial differential equation not varying under an arbitrary displacement along the rod.

As the period of oscillations and their wavelengths will decrease, the error of the approximate solution obtained in this way will increase. Another limiting case for the same system is for the minimum possible wavelength oscillations (Fig. 1). Their form can be readily calculated and employed as a first approximation in the study of the short-wave oscillations of the system. In this case the desired solution should have the form of the product of the solutions of the limiting case which is deduced from the partial differential equation.

The method of transition from the discrete models to the continuous ones has found extensive applications in physics, and the entire mechanics of the continuum is based essentially on this method.

This is not always so, however, as the case under consideration. Fluids, say, do not lend themselves in the purpose of defining a periodic equilibrium structure in reference to which oscillations are executed. Nevertheless, at a macroscopic level we perceive fluid flow as continuum flow which can be simulated by a continuous fluid model. It is true that the continuity is provided by the averaging of small-scale (microscopic) movements. The consequences of such an averaging will be discussed below. This will show how the transition to the differential equations of hydrodynamics becomes possible.

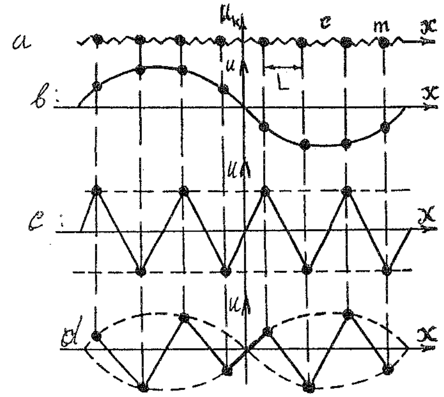

Fig.1

a- endless chain; b- long-wave oscillations; c-minimum possible wavelength oscillations; d-short-wave oscillations

In conclusion let us quote from Erwin Schrödinger who figuratively explains the efficiency of the method: "Let's assume we would tell an ancient Greek that the individual particle path in a fluid could be traced. The ancient Greek would not believe that man's limiting intellect could solve such an intricate problem. The point is that we have learned to master the whole of the process using but a single differential equation" [17].

5. Averaging

In many physical problems, some variables vary slowly, others fast. It is natural to bring out a question whether it is appropriate to have first studied the global structure under consideration, digressing from its local distinctive features, and then to investigate the system locally. It is the averaging method that is aimed to the division of the fast and slow components of the solution. Without going into the details of the method – the more because it has at present a lot of modifications – it will be noted only that it involves the introduction of the "slow" (macroscopic) and the "rapid" (microscopic) variables whose equations are separated and can be solved independently, or sequentially.

This method was developed for and gained wide use in solving problems in celestial mechanics and the nonlinear oscillation theory that are defined by the common differential equations. At present, the method is used to great advantage for solving variable-coefficient partial differential equations in such disciplines as the theory of composites, or the design of reinforced, corrugated, perforated, etc., shells [16]. An original non-homogeneous medium or structure is reduced to a homogeneous one (generally speaking, to an isotropic one) with some effective characteristics. The averaging method allows not only the obtaining of effective characteristics but also the investigation of non-homogeneous distribution of mechanical stresses in different materials and structures which is of great significance for evaluating their strength [18].

6. Renormalization

Regrettably the simple averaging of small-scale movements is not always applicable, either. There occur such problems wherein several different-scale movements show up markedly even at the macroscopic level. Among these is, for example, the study of what is known as critical phenomena related to phase transitions, or the study of turbulence. In this case a number of successive averaging procedures for all scales has to be carried out. This is the very essence of the renormalization procedure which forms the renormalization group method. A rigorous realization of the procedure, however, involves considerable technical difficulties. A practical solution of the problem is offered by a quite unexpected asymptotic method.

The fact is that in a four-dimensional imaginary world these problems do not occur, and this makes it possible to carry out an ordinary averaging. This case could be considered as a limiting one, and the quantity $\epsilon = 4 - d$ (where d is the spatial dimensionality) – as a small parameter. In the real three-dimensional world $d = 3$, and $\epsilon = 1$, which is not small. Nevertheless, an asymptotic expansion in the parameter proved to be quite effective in solving the most complicated problems of the critical-phenomena physics [19].

7. Localization

Real system deviations from a limiting (i.e., ideal) one may be of a different nature. Sometimes these deviations are small over the entire range of the system parameter variations: it is not infrequent, however, that the deviations are high, although localized within a small region. This is true for the above instance of a body streamlined by a fluid. Another example is the transition (reducing??) from the three-dimensional model of an elastic body to the two-dimensional model (plates, shells), or to the one-dimensional model (rods, beams). In this case near the body boundaries exist a narrow boundary layer (of the order of the plate or the shell wall thickness; or of cross-sectional characteristic size of the rod or the beam) wherein the three-dimensionality of the original problem manifests itself. Upon reducing the three-dimensional problem to the two-dimensional one, it is still possible to isolate the so-called end effects concentrated at shell boundaries or its structural inhomogeneities. the concept of the boundary layer is closely related to the so-called St. Venant principle that says that in the analysis of a structure it is possible to digress from the detailed load distribution pattern in fixing its elements. In actual fact, however, the distribution pattern is essential, but within narrow zones only of which the extension is defined by the element cross-sectional characteristic sizes or by the load-variation period. Mathematically, defining a boundary layer is due to the fact that a simplified differential equation is of a lesser order than an original one. The asymptotic in this case is termed singular.

8. Linearization

If the equations of a physical theory are nonlinear, then even a small number of degrees of freedom or a localized solution do not assure the overcoming of mathematical difficulties. The problem is solved by the linearization – an asymptotic method – that relies on the concept of low-intensity processes.

A linear approach (to the problem) allows formulation of such fundamental concepts as the normal vibration, the eigenfunction and the spectrum. For a linear system with the n degrees of freedom with no dumping one can always choose such "normal" coordinates which describe the system by n oscillation for pendulums not linked to one another. In other words, any motion of a linear system is represented by a linear combination of normal oscillations (or waves), that is, by the so-called expansion in a Fourier series.

It is of fundamental importance that the oscillations have been singled out not only mathematically but also physically, Thus, it is precisely the normal oscillations that will resonate under the influence of the periodic external force.

If we consider a linear system as a first approximation to a nonlinear one (that is the crux of local linearization) then, when taking into account the nonlinear corrections in the equations of a second and following approximations, there appear dummy external loads that bring about the normal oscillation resonances. This can be avoided by "touching up" the parameters of the normal linear oscillation.

However, the nonlinear systems, specially the high dimensional ones, quite often do not lend themselves to correct description in approximation of the local linearization method. Thus the combination of a high dimensionality with a strong nonlinearity was until recently considered an insurmountable difficulty in carrying out a structural study of a physical system. But a fairly extensive class of multidimensional nonlinear systems that permit such a study has been recently discovered. These systems known as the "integrable systems" have particular solutions as stable solitary waves – solitons – that are in a way analogues of normal oscillations defined in linear systems. Thus a nonlinear generalization of the Fourier method – the method of the inverse scattering problem – wherein solitons play a fundamental role taking the place of the usual Fourier components. The method of the inverse problem of scattering can be treated as the nonlocal linearization of an original nonlinear equation. In other words, the latent instability of a nonlinear system makes it possible to find a transformation that reduces the construction of an extensive class of solutions to the analysis of linear equations.

The integrable systems can in their turn act as an approximation in the analysis of the systems that approximate them, but are non-integrable within the framework of an asymptotic approach.

9. Pade approximation

So far we have assured ourselves that practically any physical problem, whose parameters include the variable parameter ϵ, can be approximately solved as ϵ approaches zero or infinity. How this "limiting" information can be used in the study of a system at the intermittent values of ϵ, say, $\epsilon = 1$? This problem is one of the most complicated in asymptotic analysis. As yet there is no general answer to the tricky question of how far the parameter ϵ can be considered small (or large) in the problem involved. Though, in many instances this problem is alleviated by the so-called two-point Pade approximants.

The notion of two-point Pade-approximant is defined [20] as follows. Let

$$F(\epsilon) = \sum_{i=0}^{\infty} \alpha_i \epsilon^i \qquad \text{when } \epsilon \to 0 \tag{9.1}$$

$$F(\epsilon) = \sum_{i=0}^{\infty} \beta_i \epsilon^{-i} \qquad \text{when } \epsilon \to \infty \tag{9.2}$$

The two-point Pade-approximant is represented by the function

$$F(\epsilon) = \left(\sum_{k=0}^{m} \alpha_k \epsilon^k \right) \left(1 + \sum_{k=0}^{m} \alpha_k \epsilon^k \right)^{-1}$$

in which $m + 1$ coefficients of expansion in the Taylor series when $\epsilon \to 0$, and m coefficients in the Lorentz series when $\epsilon \to \infty$ coincide with the corresponding coefficients of the series (9.1) and (9.2).

The practice shows that the Pade approximants do indeed quite often allow the limiting expansions to be "sewed together" after defining the regions of "small" and "large" values of ϵ. This resembles a known interpolation procedure, that is, the reconstruction of the intermediate values of a quantity by its two extreme values. The role of such known values is played in this case by the asymptotics as ϵ tends to zero and to infinity.

For instance, for equation (2.2) the Pade approximant for the first root

$$y = (2 + 0.57\epsilon + 0.12\epsilon^2)(1 + 0.12\epsilon)^{-1}$$

derived on the basis of the asymptotics of the form (2.5) as ϵ tends to zero or or the form (2.6) as ϵ tends to infinity, defines satisfactorily the exact solution at any value of ϵ (Fig. 2).

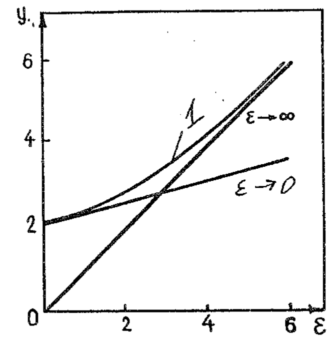

Fig.2

1-exact solution and two-point approximants

10. Modern computers and asymptotic methods

The reader must have repeatedly asked himself a question: are the asymptotic methods of any practical use at all when there are computers? Is it not simpler to write a programme for any original problem to solve it numerically by using standard procedures?

The answer may be like this. First the asymptotic methods are very useful in the preliminary stage of solving a problem even in cases where the principal aim is to obtain the numerical results. The asymptotic analysis makes it possible to choose the best numerical method and gain an understanding of a vast body of numerical material, though not properly arranged. Secondly, the asymptotic methods are specially effective in those regions of parameter values where machine computations are faced with serious difficulties. Laplace used to say not without reason that the asymptotic methods are "more accurate, the more they are needed". Moreover, the possibility exists of developing such algorithms wherein smooth portions of solutions are obtained numerically, and the asymptotic approaches are applied to that parameter value regions where these solutions change drastically, say, within boundary layers. Thirdly, the asymptotic methods develop our intuition in every possible way and play, as noted above, an important role in shaping the mentality of, say, a contemporary scientist or engineer. Therefore, it would be more proper to consider the asymptotic and numerical methods not as competing, but as mutually complementary ones.

Again computers further considerably the development of the asymptotic method. For instance, defining higher approximations is a major difficulty in applying the asymptotic methods. In solving complex problems by manual calculations one may succeed in defining two or three approximations at the most. Now the burden of manual calculations can be shouldered by the computer.

11. Asymptotic methods and teaching physics

"Few of the equations of physics have exact solutions which are manageable, and one usually has to have recourse either to approximate methods or to numerical solutions. Numerical work becomes cumbersome if the problem has a great number of variables, or if one is interested in a general survey of possible solutions. In those cases the natural approach is by approximation. In teaching physics we probably overemphasize the exceptional problems which have closed solutions in terms of elementary functions, and do not give enough attention to the more common situation in which approximations have to be used. Beginners are usually uncomfortable with approximations, and, even if only an approximate answer is required, often prefer to find the exact answer, if this is possible, and then to approximate. This is understandable because the art of choosing a suitable approximation, of checking its consistency (e.g. ensuring there are no cancellations) and finding at least intuitive reasons for expecting the approximation to be satisfactory, is much more solving an equation exactly" [21].

References

1. A. H. Nayfeh, *Perturbations methods*, New York: Wiley, 1973.
2. A. H. Nayfeh, *Introduction to perturbation techniques*, New York: Wiley, 1981.
3. M. Van Dyke, *Perturbation methods in fluid mechanics*, New York: Academic Press, 1964.
4. J. Kevorkian and J. D. Cole, *Perturbation methods in applied mathematics*, N.J., 1981.
5. S. S. Lin and L. A. Segel, *Mathematics applied to deterministic problems in the natural sciences*, Philadelphia: SIAM, 1988.
6. J. D. Cole, *Perturbation methods in applied mathematics*, Boston: Ginn (Blaisdell), 1968.
7. R. E. O'Mally, *Introduction to singular perturbation*, New York: Academic Press, 1974.
8. M. D. Kruskal, *Asymptotology*, *Mathematical models in physical sciences*, New Jersey: Prentice-Hall, 1963. pp 17-48.
9. K. O. Friedrichs, *Asymptotic phenomena in mathematical physics*, Bull. Amer. Math. Soc. **61** (1955) 485-504.
10. L. A. Segel, *The importance of asymptotic analysis in applied mathematics*, Amer. Math. Monthly **73** (1966) 7-14.
11. R. G. Barabtsev, *Asymptotic versus classical mathematics*, Topics in Mathematical Analysis (1989) 49-64.
12. V. M. Babich and V. S. Buldirev, *The art of asymptotics*, Vestnik Leningrad Univ. Math. **10** (1982) 227-335.
13. I. V. Andrianov and L. I. Manevich, *Two hiepostaces of asymptotics*, Priroda, No.4 (1987) 85-97 (in Russian).
14. I. V. Andrianov and L. I. Manevich, *Asymptotic methods and physical theories*, Moscow: Znanie, 1989 (in Russian).
15. I. V. Andrianov, I. F. Obraztsov and B. V. Nerubaylo, *Asymptotic methods in the structural mechanics of thin-walled structures*, Moscow: Mashinostroyenie, 1991 (in Russian).
16. I. V. Andrianov, V. A. Lesnichaya and L. I. Manevich, *Homogenization methods in statics and dynamics of reinforced shells*, Moscow: Nauka, 1985 (in Russian).
17. E. Schrödinger, *Über Indeterminismus in der Physik. Ist die Naturwissenschaft milieubedingst? Zwei Vortäge zur Kritik der wissenschaftlichen Erkenntnis*, Leipzig, 1932. pp. 25-62.
18. N. Bakhvalov and G. Panasenko, *Averaging processes in periodic media. Mathematical problems in mechanics of composite materials.* Dordrecht: Kluwer, 1989.

19. K. Wilson, *Problems in physics with many seals of length*, Scientific American, No.8 (1979) 140-157.

20. G. A. Baker and P. Graves-Morris, *Pade approximants. Part 1: Basic theory. Part 2: Extensions and applications*, New York: Addison-Wesley, 1981.

21. R. Peierls, *Model-making in physics*, Contemp. Phys. 21 (1980) 3-17.

DNEPROPETROVSK CIVIL ENGINEERING INSTITUTE, UKRAINE, 320005, DNEPROPETROVSK CHERNYSHEVSKOGO 2Ya.

INSTITUTE OF CHEMICAL PHYSICS, MOSCOW, RUSSIA, 117977, KOSYGINA 4.

RECEIVED BY THE EDITOR: APRIL 11, 1992.

Index

VOLUME 2

Almeida, D. F., The First Mathematical Proof of Wigner's Theorem *(With C. S. Sharma)* 113

Andrianov, I. V., Asymptotology I: Problems, Ideas and Results *(With L. I. Manevich)* 137
Application of *P*-Invertibility of Matrices, An, *Anthea Grubb* 107
Asymptotology I: Problems, Ideas and Results, *I. V. Andrianov and L. I. Manevich* 137
Automorphisms of Self-adjoint Operators in a Unitary Space of Dimension Two, Eigenvalue Preserving, *Anthea Grubb* 101
Daglish, George, The Essential Mathematics of Remote Sensing, 129
Domain, A Question of , *C. S. Sharma* 125
Eigenvalue Preserving Automorphisms of Self-adjoint Operators in a Unitary Space of Dimension Two, *Anthea Grubb* 101
Essential Mathematics of Remote Sensing, The, *George Daglish* 129
First Mathematical Proof of Wigner's Theorem, The, *D. F. Almeida and C. S. Sharma* 113
Grubb, Anthea, An Application of *P*-Invertibility of Matrices 107
Grubb, Anthea, Eigenvalue Preserving Automorphisms of Self-adjoint Operators in a Unitary Space of Dimension Two 101
Manevich, L. I., Asymptotology I: Problems, Ideas and Results *(With I. V. Andrianov)* 137
Mathematical Proof of Wigner's Theorem, The First, *D. F. Almeida and C. S. Sharma* 113
Mathematics of Remote Sensing, The Essential, *George Daglish* 129
Matrices, An Application of *P*-Invertibility of, *Anthea Grubb* 107
P-Invertibility of Matrices, An Application of, *Anthea Grubb* 107
Question of Domain, A, *C. S. Sharma* 125
Remote Sensing, The Essential Mathematics of, *George Daglish* 129
Self-adjoint Operators in a Unitary Space of Dimension Two, Eigenvalue Preserving Automorphisms of, *Anthea Grubb* 101
Sharma, C. S., A Question of Domain, 125
Sharma, C. S., The First Mathematical Proof of Wigner's Theorem *(With D. F. Almeida)* 113
Wigner's Theorem, The First Mathematical Proof of, *D. F. Almeida and C. S. Sharma* 113